Generation conditions for an acoustic laser

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A theoretical scheme of an acoustic laser is studied. A liquid dielectric with dispersed particles is considered as an active medium. The generation conditions for this acoustic laser are evaluated. It is shown that two types of losses must be overcome in order for generation to begin. The first type results from the energy dissipation in the active medium and the second one is caused by radiation losses at the boundaries of the resonator.

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Recently, a theoretical scheme for an acoustic laser was proposed in Ref. [1]. A liquid dielectric with dispersed particles was considered as an active medium. The pumping was created by an oscillating electric field deforming dispersed particle volumes. The phase bunching of the initially incoherent emitters was realized by acoustic radiative forces. The suggested scheme is an analog of the free-electron laser (FEL). Different types of oils or distilled water can serve as a liquid dielectric with gas bubbles as dispersed particles.

The purpose of this paper is to evaluate the generation conditions for this acoustic laser. It is shown that two types of losses must be overcome for generation to begin. The first type results from the energy dissipation in the active medium. The second type is caused by radiation losses at the boundaries of the resonator.

The theoretical scheme for the acoustic laser is shown in Fig. 1. The active medium is placed between the two semi-infinite walls.

The equations describing the interaction of a sound wave with the active medium are of the form [1]

$$\left[\Delta-\frac{1}{c_l^2}\frac{\partial^2}{\partial t^2}-(\alpha_0+i\beta_0)\right]\Psi(\mathbf{r},t)=(\alpha'+i\beta')FP_E\exp(i\omega t)\ , \tag{1}$$

$$\gamma\frac{\partial n'}{\partial t}-n_0\left[(\alpha_0-i\beta_0)P_0^*\Delta\Psi+(\alpha_0+i\beta_0)P_0\Delta\Psi^*\right]=0\ . \tag{2}$$

Here $\Psi(\mathbf{r},t)$ is the useful acoustic mode being excited in the resonator; P_E is the pumping pressure amplitude which may be considered to be independent of the spatial coordinates; ω is the angular frequency of the pumping pressure; n_0 is the initial number of particles per unit liquid volume (to simplify the calculations, with all particles assumed to be equal in radius); $n'=n'(\mathbf{r},t)$ is the deviation of the distribution function of the dispersed particles from the initial equilibrium value n_0; $\gamma=12\pi\mu_l R_0\rho_l\omega^2 n_0 f_v/f_r$ is the constant factor (μ_l is the liquid viscosity, ρ_l is the liquid density, R_0 is the mean particle radius, and f_v and f_r are the correcting factors); $\alpha_0=-4\pi n_0\mathrm{Re}A$ and $\beta_0=-4\pi n_0\mathrm{Im}A$ are the constant real terms, where A is the scattering amplitude of the sound wave by one particle [in the case of a liquid with gas bubbles, $A=R_0/(\omega_0^2/\omega^2-1+i\delta)$, ω_0 is the resonance frequency of the bubble, and δ is the absorption constant]; the deviations $\alpha'=-4\pi n'\mathrm{Re}A$, and $\beta'=-4\pi n'\mathrm{Im}A$ are caused by the change of number of particles per unit liquid volume due to the acoustic radiation forces; $F=[1-(\alpha_0+i\beta_0)/k_l^2]^{-1}$, where $k_l=\omega/c_l$ is the wave number in the pure liquid; and $P_0(t)=FP_E\exp(i\omega t)$.

Let us assume that particles are far from resonance ($\beta_0\ll|\alpha_0|$), and the particle content is small, i.e., $|\alpha_0|\ll k_1^2$. In order to solve Eqs. (1) and (2), we use the substitution $\Psi(\mathbf{r},t)=-[\partial Z(\mathbf{r},t)/\partial t]\exp(i\omega t)$. As a result, for Z the differential equation of the third order, we have

$$\left[\Delta+\frac{\omega^2}{c_l^2}-\frac{2i\omega}{c_l^2}\frac{\partial}{\partial t}-\frac{1}{c_l^2}\frac{\partial^2}{\partial t^2}-(\alpha_0+i\beta_0)\right]\frac{\partial Z}{\partial t}=\frac{\alpha_0^2P_E^2}{\gamma}\Delta(Z+Z^*)\ . \tag{3}$$

We restrict our consideration to the simplest case when

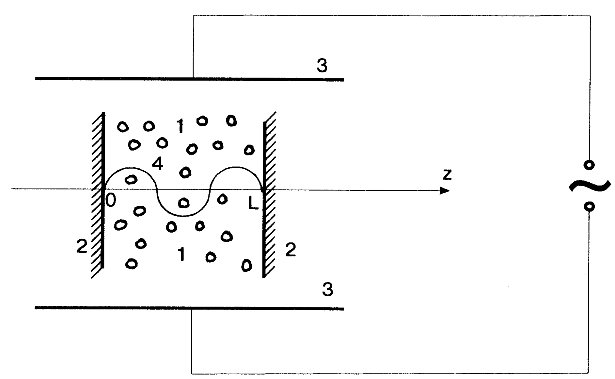

FIG. 1. The scheme for an acoustic laser. (1) The active medium. (2) Solid walls of the resonator. (3) Electromagnetic system creating the periodic electric field. (4) An acoustic mode.

Z is a function of one spatial coordinate only, z, for example. If the pumping is absent ($P_E=0$), we seek a solution of this equation as $Z\sim\exp(i\omega' t-ikz)$. Then we can obtain the characteristic equation

$$k^2=\frac{(\omega+\omega')^2}{c_l^2}-\alpha_0-i\beta_0\ , \tag{4}$$

This equation corresponds to the decaying (in the z direction) wave with the angular frequency $(\omega+\omega')$.

Pumping results first in the appearance of a back wave with the angular frequency $(\omega-\omega')$, and second in the dynamical instability of the system being considered. Let us seek a solution of Eq. (3) in a form that follows $Z=A\exp(i\omega' t-ikz)+\beta\exp(-i\omega' t+ik^*z)$. Then one can obtain

$$\left[\frac{(\omega+\omega')^2}{c_l^2}-k^2-\alpha_0-i\left[\beta_0+\frac{\alpha_0^2P_E^2k^2}{\gamma\omega'}\right]\right]A-i\frac{\alpha_0^2P_E^2k^2}{\gamma\omega'}B^*=0\ , \tag{5}$$

$$-i\frac{\alpha_0^2P_E^2k^2}{\gamma\omega'}A+\left[\frac{(\omega-\omega')^2}{c_l^2}-k^2-\alpha_0+i\left[\beta_0-\frac{\alpha_0^2P_E^2k^2}{\gamma\omega'}\right]\right]B^*=0\ . \tag{6}$$

Assuming that the amplitude of the back wave is small, i.e., $|B|\ll|A|$, we can find

$$k^2\approx\frac{(\omega+\omega')^2}{c_l^2}-\alpha_0-i\left[\beta_0+\frac{\alpha_0^2P_E^2(\omega+\omega')^2}{\gamma\omega' c_l^2}\right]\ , \tag{7}$$

$$B^*=\sigma^*(\omega')A\approx\frac{i\dfrac{\alpha_0^2P_E^2(\omega+\omega')^2}{\gamma\omega' c_l^2}}{\left[-\dfrac{4\omega\omega'}{c_l^2}+2i\beta_0\right]}A\ . \tag{8}$$

Note that the waves propagating in the z direction [with angular frequencies $(\omega+\omega')$] are amplified if $\omega'<0$, and the condition

$$P_E\geq P_{\rm st}=\frac{c_l}{|\alpha_0|(\omega+\omega')}\sqrt{\gamma\beta_0|\omega'|} \tag{9}$$

holds true. The quantity $P_{\rm st}$ corresponds to the starting current in the FEL [2].

The relation $|B|\ll|A|$ [or $|\sigma(\omega')|\ll 1$] is valid when the deviation ω'/ω is not too small, i.e.,

$$\frac{\omega|\omega'|}{c_1^2}\gg\beta_0\ ,\quad \frac{\alpha_0^2P_E^2(\omega+\omega')^2}{\gamma|\omega'|c_l^2}\ . \tag{10}$$

In the general case we have to solve the characteristic equation for the set of equations (5) and (6), which is obtained by setting the determinant of the latter set equal to zero:

$$k^2\approx\frac{\omega^2+\omega'^2}{c_l^2}-\alpha_0-i\frac{\alpha_0^2P_E^2(\omega+\omega')^2}{\gamma\omega'}+\left[\left[\frac{2\omega\omega'}{c_l^2}-i\beta_0\right]^2-\left[\frac{\alpha_0^2P_E^2(\omega+\omega')^2}{\gamma\omega'}\right]^2\right]^{1/2}\ . \tag{11}$$

In order to limit the quantity k^2 at $\omega'/\omega\to 0$, we take the plus radical sign in the right-hand side of Eq. (11).

The relation $\sigma(\omega')=B/A^*$ can be determined from Eq. (6).

In order to reveal the generation conditions for the acoustic laser, let us consider its operation in the regime of amplification, i.e., the passage of a signal wave from $z<0$ to $z>L$ through the active medium $(0\leq z\leq L)$ (see Fig. 1).

Let $\Psi_{\rm int}=A_{\rm int}\exp[i(\omega+\omega')t-ik_s(\omega+\omega')z]$ be an incident wave, where $k_s(\omega+\omega')=(\omega+\omega')/c_s$, and c_s is the velocity of the sound in solid walls.

As has been shown above, acoustic waves being excited in the active medium have the angular frequencies $(\omega+\omega')$ and $(\omega-\omega')$. Consequently, two reflected waves [with frequencies $(\omega+\omega')$ and $(\omega-\omega')$, respectively] should be presented in the range $z<0$. Thus in the range $z<0$ the resulting pressure wave is of the form

$$\begin{aligned}\Psi(z<0)=&A_{\rm int}\exp[i(\omega+\omega')t-ik_s(\omega+\omega')z]\\&+R_s^{(+)}\exp[i(\omega+\omega')t+ik_s(\omega+\omega')z]\\&+R_s^{(-)}\exp[i(\omega-\omega')t+ik_s(\omega-\omega')z]\ .\end{aligned} \tag{12}$$

Let us calculate the pressure wave in the active medium as

$$\begin{aligned}\Psi(0\leq z\leq L)=&A^{(+)}\exp[i(\omega+\omega')t-ik(\omega+\omega')z]+B^{(+)}\exp[i(\omega-\omega')t+ik^*(\omega+\omega')z]\\&+C^{(+)}\exp[i(\omega+\omega')t+ik(\omega+\omega')z]+D^{(+)}\exp[i(\omega-\omega')t-ik^*(\omega+\omega')z]\\&+A^{(-)}\exp[i(\omega-\omega')t-ik(\omega-\omega')z]+B^{(-)}\exp[i(\omega+\omega')+ik^*(\omega-\omega')z]\\&+C^{(-)}\exp[i(\omega-\omega')t+ik(\omega-\omega')z]+D^{(-)}\exp[i(\omega+\omega')t-ik^*(\omega-\omega')z]\ .\end{aligned} \tag{13}$$

Then the pressure wave in the range $z>L$ takes the form

$$\Psi(z>L)=T^{(+)}\exp[i(\omega+\omega')t-ik_s(\omega+\omega')z]+T^{(-)}\exp[i(\omega-\omega')t-ik_s(\omega-\omega')z]\ . \tag{14}$$

The boundary conditions at $z=0$ and L are determined by the equality of pressures and velocities of media [the velocity of a medium is $\mathbf{V}\sim(1/\rho)\nabla\Psi$]. It leads to a system of eight equations for 12 complex coefficients $R_s^{(\pm)}$, $A^{(\pm)}$, $B^{(\pm)}$, $C^{(\pm)}$, $D^{(\pm)}$, and $T^{(\pm)}$ which are connected by four additional equations: $B^{(\pm)}=\sigma(\pm\omega')A^{(\pm)*}$ and $D^{(\pm)}=\sigma(\pm\omega')C^{(\pm)*}$.

Generation conditions for the acoustic laser correspond to zero of the determinant of these equations. If relations (10) hold true (or $|\sigma(\pm\omega')| \ll 1$), this condition takes the form

$$\left[\frac{k(\omega+\omega')}{\rho_l}+\frac{k_s(\omega+\omega')}{\rho_s}\right]^2 \exp[ik(\omega+\omega')L]-\left[\frac{k(\omega+\omega')}{\rho_1}-\frac{k_s(\omega+\omega')}{\rho_s}\right]^2 \exp[-ik(\omega+\omega')L]=0 , \quad (15)$$

where ρ_l is the liquid density, and ρ_1 is the solid boundary density.

Using (7), one obtains the solution of Eq. (15) at $\omega' < 0$:

$$k_{\text{eff}}L=\left[\frac{(\omega+\omega')^2}{c_l^2}-\alpha_0\right]^{1/2} L=\pi n \quad (n=1,2,3\cdots) , \quad (16)$$

$$\frac{\alpha_0^2 P_E^2(\omega+\omega')^2}{\gamma|\omega'|c_l^2}=\beta_0-\frac{k_{\text{eff}}}{L}\ln\frac{1-\dfrac{\rho_l c_l}{\rho_s c_s}}{1+\dfrac{\rho_l c_l}{\rho_s c_s}} \approx \beta_0+\frac{2k_{\text{eff}}}{L}\frac{\rho_l c_l}{\rho_s c_s} . \quad (17)$$

The physical meaning of generation conditions (16) and (17) is clear. Equation (16) shows that the acoustic mode can be excited only if L contains an integral number of half wavelengths. The second condition gives the starting quantity of P_{st}. Two types of losses must be overcome for the beginning of a generation. The first type is conditioned by the energy dissipation in the active medium [the first term on the right-hand side of Eq. (17)]. The second type is caused by the radiation losses at the boundaries of the resonator [the second term on the left-hand side of Eq. (17)]. Usually this second term is inversely proportional to the volume of the stored energy (see, for example, [3]), i.e., L^{-1}.

The starting pressure P_{st} increases with an increase of the liquid viscosity $\mu_l \sim \gamma$, which prevents dispersed particles from bunching.

The obtained results can slightly improve the estimation of P_{st} given in [1]. If the resonator length is sufficiently large, we have

$$P_{\text{st}} \approx \frac{c_l}{|\alpha_0|}\left[\frac{\beta_0\gamma|\omega'|}{(\omega+\omega')^2}\right]^{1/2} . \quad (18)$$

For a liquid dielectric with gas bubbles in the limit $\omega \ll \omega_0$, we can obtain

$$P_{\text{st}} \approx c_l\sqrt{\mu_l\rho_l\delta|\omega'|}/(1+\omega'/\omega) . \quad (19)$$

In the case of air bubbles with radii $R_0=15$ μm in distilled water, at the frequency $f=\omega/2\pi=1.5$ kHz and the ratio $\omega'/\omega \approx -0.1$, we find $P_{\text{st}}=0.46$ kPa. It corresponds (see [1]) to the electric intensity $E_0 \approx 10$ kV/cm.

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[3] S. Kino, *Acoustic Waves* (Prentice-Hall, Englewood Cliffs, NJ, 1987).